Sec 13 رياضيات

Gamma Function

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x}\, dx$$

① $\Gamma(1) = 1$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

② $\Gamma(n) = (n-1)!$

③ $\Gamma(x)\Gamma(1-x) = \dfrac{\pi}{\sin(\pi x)}$

④ $\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$

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① α عدد سالب $\quad \Gamma(\alpha) = \dfrac{\Gamma(\alpha+1)}{\alpha}$

② n عدد صحيح موجب $\quad \Gamma(n) = (n-1)!$

③ α عدد كسر موجب $\quad \Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$

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* $\Gamma(9) = 8!$
* $\Gamma(\frac{5}{2}) = \frac{3}{2}\cdot\frac{1}{2}\,\Gamma(\frac{1}{2}) = \frac{3}{4}\sqrt{\pi}$
* $\Gamma(\frac{-5}{2}) = \frac{-2}{5}\cdot\frac{-2}{3}\cdot\frac{-1}{2}\cdot\Gamma(\frac{1}{2})$
* $\Gamma(\frac{1}{3})\,\Gamma(\frac{2}{3}) = \dfrac{\pi}{\sin(\frac{\pi}{3})}$

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① $e^{f(x)}$ نعوض عن $f(x)$ بـ $-u$

② $a^{f(x)} = e^{f(x)\ln a}$

③ $\ln(x) \leftarrow \ln(x) = -t$

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① $\int_0^{\infty} x^3 e^{-x}\, dx = \Gamma(4) = 3!$

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② $\int_0^{\infty} \sqrt{x}\; e^{-x^3}\, dx$

let $x^3 = u \rightarrow x = u^{\frac{1}{3}}$

$dx = \frac{1}{3} u^{\frac{-2}{3}}\, du$

$\frac{1}{3}\int_0^{\infty} u^{\frac{1}{6}} \cdot u^{\frac{-2}{3}}\, e^{-u}\, du$

$= \frac{1}{3}\int_0^{\infty} u^{\frac{-3}{6}}\, e^{-u}\, du = \frac{1}{3}\Gamma(\frac{-3}{6})$

$= \frac{1}{3}\Gamma(\frac{-1}{2}) = \frac{1}{3}\cdot\frac{-2}{1}\cdot\sqrt{\pi}$

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③ $\int_0^{\frac{1}{2}} x^{m-1} \ln\left(\frac{1}{2x}\right) dx$

$= -\int_0^{\frac{1}{2}} x^{m-1} \ln(2x)\, dx$

let $\ln(2x) = -t$

$x = \frac{1}{2} e^{-t}$

$dx = \frac{-1}{2} e^{-t}\, dt$

at $x = 0 \rightarrow t = \infty$

at $x = \frac{1}{2} \rightarrow t = 0$

$= -\int_{\infty}^{0} (\frac{1}{2})^{m-1} e^{-(m-1)t} \cdot (-t)\left(\frac{-1}{2}\right) e^{-t}\, dt$

$= +(\frac{1}{2})^m \int_0^{\infty} t\; e^{-mt}\, dt$

let $u = mt$

$dt = \frac{1}{m}\, du$

$= (\frac{1}{2})^m \int_0^{\infty} \frac{1}{m} u \cdot e^{-u} \cdot \frac{1}{m}\, du$

$= (\frac{1}{2})^m \cdot \frac{1}{m^2}\, \Gamma(2)$

$= (\frac{1}{2})^m \cdot \frac{1}{m^2}$

Beta Function

① $\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx$

② $\beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\, \cos^{2n-1}\theta\, d\theta$

③ $\beta(m,n) = \int_0^{\infty} \dfrac{y^{m-1}}{(1-y)^{m+n}}\, dy$

④ $\beta(m,n) = \beta(n,m)$

⑤ $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Ex $\int_0^1 x^5 (1-x)^8 dx$

$= \beta(6, 9) = \frac{\Gamma(6)\Gamma(9)}{\Gamma(10)}$

$\int_0^1 \frac{5x}{\sqrt{1-x^5}} dx$

Let $x^5 = t \rightarrow x = t^{\frac{1}{5}}$

$dx = \frac{1}{5} t^{\frac{-4}{5}}$

$\frac{1}{5}\int_0^1 \frac{t^{-\frac{4}{5}+\frac{1}{5}}}{\sqrt{1-t}} dt = \frac{1}{5}\int_0^1 t^{\frac{-3}{5}} (1-t)^{\frac{-1}{2}} dt$

$= \beta\left(\frac{2}{5}, \frac{1}{2}\right) = \frac{\Gamma(\frac{2}{5})\Gamma(\frac{1}{2})}{\Gamma(\frac{9}{10})}$

$\int_0^{\frac{\pi}{2}} \sqrt{\frac{\sin^3\theta}{\cos\theta}} d\theta = \int_0^{\frac{\pi}{2}} \sin^{\frac{3}{2}}\theta \cos^{\frac{-1}{2}}\theta \, d\theta$

$= \frac{1}{2}\beta\left(\frac{5}{4}, \frac{1}{4}\right) = \frac{\Gamma(\frac{5}{4})\Gamma(\frac{1}{4})}{\Gamma(\frac{6}{4})}$

Bessel function

$x^2y'' + xy' + (x^2 - n)y = 0$

$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (\frac{x}{2})^{2r+n}}{r! \, (r+n+1)}$

$\rightarrow n_1 - n_2 =$ non Integer

$y_{G.S} = C_1 J_n(x) + C_2 J_{-n}(x)$

$\rightarrow n_1 - n_2 =$ Integer

$y_{G.S} = C_1 J_n(x) + C_2 y_n(x)$

$y_n(x) = \lim_{n \to n} \frac{J_n(x)\cos(n\pi) - J_{-n}(x)}{\sin(n\pi)}$